COMMUTATIVE C^*-ALGEBRAS OF TOEPLITZ OPERATORS ON COMPLEX PROJECTIVE SPACES

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Abstract. We prove the existence of commutative C^*-algebras of Toeplitz operators on every weighted Bergman space over the complex projective space $\mathbb{P}^n(\mathbb{C})$. The symbols that define our algebras are those that depend only on the radial part of the homogeneous coordinates. The algebras presented have an associated pair of Lagrangian foliations with distinguished geometric properties and are closely related to the geometry of $\mathbb{P}^n(\mathbb{C})$.

1. Introduction

The existence of nontrivial commutative C^*-algebras of Toeplitz operators on bounded domains has shown to be a remarkably interesting phenomenon (see [2], [8], [9], [10], [13]). Large families of symbols defining such commutative algebras on every weighted Bergman space have been proved to exist on the unit ball and Reinhardt domains. Also, in [7] it was proved the existence of commutative C^*-algebras of Toeplitz operators for the sphere. All of the examples exhibited in these references come equiped with a distinguished geometry and can be described in terms of the isometry group of the domain.

In this work we study the complex projective space $\mathbb{P}^n(\mathbb{C})$ and its hyperplane line bundle H. It is well known that these objects provide the usual setup to consider quantization through the use of weighted Bergman spaces (see [11]). Note that, due to the compactness of $\mathbb{P}^n(\mathbb{C})$, the weight of the Bergman spaces has a discrete set of values. Within this setup for $\mathbb{P}^n(\mathbb{C})$, we prove the existence of commutative C^*-algebras of Toeplitz operators on all weighted Bergman spaces. As in the case of previous works, our algebras can be described in terms of the symmetries of $\mathbb{P}^n(\mathbb{C})$. To elaborate on this, recall that for every hyperplane $P \subset \mathbb{P}^n(\mathbb{C})$, the open set $\mathbb{P}^n(\mathbb{C}) \setminus P$ can be canonically identified with the complex plane $\mathbb{C}^n$. With respect to such identification, we prove that the symbols that depend only on the radial components of the coordinates of $\mathbb{C}^n$ define commutative C^*-algebras of Toeplitz operators on every weighted Bergman space over $\mathbb{P}^n(\mathbb{C})$ (see Theorem 5.5); this sort of symbols are called separately radial. Furthermore, we construct unitary equivalences of the weighted Bergman spaces with suitable (finite dimensional) Hilbert spaces such that the Toeplitz operators with separately radial symbols are simultaneously turned into multiplication operators.

We also prove that our commutative C^*-algebras have a distinguished geometry that describes them. Recall that the connected component of the group of isometries

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of $\mathbb{P}^n(\mathbb{C})$ is given by the group $\mathrm{SU}(n+1)$ of $(n+1)\times(n+1)$ unitary matrices with determinant 1. It is proved that the separately radial symbols are those invariant under the action of the diagonal matrices of $\mathrm{SU}(n+1)$ (see Lemma 6.1); in particular, this provides a description of the separately radial symbols that depends on $\mathbb{P}^n(\mathbb{C})$ only, and not on a set of coordinates. Also, it allows us to show that every commutative C^*-algebra obtained from separately radial symbols has an associated pair of Lagrangian foliations with special geometric properties (see Theorem 6.11). Such properties come from the group $\mathbb{A}(n)$ of isometries of $\mathbb{P}^n(\mathbb{C})$ defined by the diagonal matrices in $\mathrm{SU}(n+1)$. It turns out that $\mathbb{A}(n)$ is a maximal Abelian subgroup of the group of isometries of $\mathbb{P}^n(\mathbb{C})$ (see Section 6). We prove that, up to an isometry, $\mathbb{A}(n)$ is the only such maximal Abelian subgroup. This behavior is in contrast with the existence of $n+2$ maximal Abelian subgroups of the group of isometries of $\mathbb{B}^n$, which provides $n+2$ nonequivalent commutative C^*-algebras of Toeplitz operators (see [8] and [9]). This is why for the projective space $\mathbb{P}^n(\mathbb{C})$ there exist, up to an isometry, only one commutative C^*-algebra of Toeplitz operators that has an associated pair of foliations as described in Theorem 6.11.

As for the contents of this work, in Section 2 we present some differential geometric preliminaries on $\mathbb{P}^n(\mathbb{C})$. Section 3 recalls the basic material on quantization on compact Kähler manifolds. Section 4 recollects the known results specific to the quantization for complex projective spaces including weighted Bergman spaces. Sections 5 and 6 contain our main results and proofs on the existence of commutative C^*-algebras and their geometric description; our techniques are closely related to those found in [8] and [9].

2. Geometric preliminaries on $\mathbb{P}^n(\mathbb{C})$

Recall that $\mathbb{P}^n(\mathbb{C})$ is the complex n-dimensional manifold that consists of the elements $[w] = \mathbb{C}w \setminus \{0\}$, where $w \in \mathbb{C}^{n+1}\setminus\{0\}$. For every $j = 0, \dots, n$, we have an open set

$$U_j = \{[w] \in \mathbb{P}^n(\mathbb{C}) : w_j \neq 0\}$$

and a biholomorphism $\varphi_j : U_j \to \mathbb{C}^n$ given by

$$\varphi_j([w]) = \frac{1}{w_j}(w_0, \dots, \widehat{w}_j, \dots, w_n) = (z_1, \dots, z_n),$$

where the numbers z_k are known as the homogeneous coordinates with respect to the map φ_j. The collection of all such maps yields the holomorphic atlas of $\mathbb{P}^n(\mathbb{C})$ and provides (equivalent) realizations of $\mathbb{C}^n$ as an open dense conull subset of $\mathbb{P}^n(\mathbb{C})$.

We refer to Example 6.3 of [5] for the details on the following construction of the Fubini-Study metric on $\mathbb{P}^n(\mathbb{C})$.

For every $j = 0, \dots, n$ consider the function $f_j : U_j \to \mathbb{C}$ given by

$$f_j([w]) = \sum_{k=0}^{n} \frac{w_k \overline{w}_k}{w_j \overline{w}_j} = 1 + \sum_{k=1}^{n} z_k \overline{z}_k, \tag{2.1}$$

for the above homogeneous coordinates z_k with respect to φ_j. Then, it is easily seen that

$$\partial\overline{\partial}\log f_j = \partial\overline{\partial}\log f_k$$

on $U_j \cap U_k$ for every $j, k = 0, \dots, n$. In particular, there is a well defined closed $(1,1)$-form ω on $\mathbb{P}^n(\mathbb{C})$ given by

$$\omega = i\partial\overline{\partial} \log f_j, \tag{2.2}$$

on U_j. This yields the canonical Kähler structure on $\mathbb{P}^n(\mathbb{C})$ for which it is the Hermitian symmetric space with constant positive holomorphic sectional curvature. The corresponding Riemannian metric is known as the Fubini-Study metric. We refer to [5] for these and the rest of the remarks in this section on the geometry of $\mathbb{P}^n(\mathbb{C})$ induced by ω.

With respect to the chart φ_0, we have the following induced Kähler form on $\mathbb{C}^n$

$$\omega_0 = (\varphi_0^{-1})^*(\omega) = i\frac{(1+|z|^2)\sum_{k=1}^n dz_k \wedge d\overline{z}_k - \sum_{k,l=1}^n \overline{z}_k z_l dz_k \wedge d\overline{z}_l}{(1+|z|^2)^2}.$$

The volume element on $\mathbb{P}^n(\mathbb{C})$ with respect to the Fubini-Study metric is defined by

$$\Omega = \frac{1}{(2\pi)^n}\omega^n.$$

The following result is a consequence of the definition of the volume element of a Riemannian metric and its properties for Hermitian symmetric spaces (see [3]).

Lemma 2.1. *The volume element on $\mathbb{C}^n$ induced by the Fubini-Study metric of $\mathbb{P}^n(\mathbb{C})$ is given by*

$$\Omega = \frac{1}{(2\pi)^n}\omega_0^n = \frac{1}{\pi^n}\frac{dV(z)}{(1+|z_1|^2+\cdots+|z_n|^2)^{n+1}}$$

where $dV(z) = dx_1 \wedge dy_1 \wedge \cdots \wedge dx_n \wedge dy_n$ is the Lebesgue measure on $\mathbb{C}^n$.

We also recall that the Hopf fibration of $\mathbb{P}^n(\mathbb{C})$ is given by

$$\begin{aligned} \pi : S^{2n+1} &\to \mathbb{P}^n(\mathbb{C}) \\ w &\mapsto [w], \end{aligned}$$

where $S^{2n+1} \subset \mathbb{C}^{n+1}$ is the unit sphere centered at the origin.

We denote with $\mathrm{SU}(n+1)$ the Lie group of $(n+1)\times(n+1)$ unitary matrices with determinant 1. In other words, for A a complex $(n+1)\times(n+1)$ matrix with $\det(A) = 1$, we have $A \in \mathrm{SU}(n+1)$ if and only if $A^*A = I_{n+1}$. For Z_{n+1} the group of $(n+1)$-th roots of unity in $\mathbb{C}$, we also consider the quotient Lie group $\mathrm{PSU}(n+1,\mathbb{C}) = \mathrm{SU}(n+1,\mathbb{C})/Z_{n+1}I_{n+1}$, and we denote with λ the natural quotient map of $\mathrm{SU}(n+1)$ onto $\mathrm{PSU}(n+1)$. It is well known that these Lie groups are connected and compact. The following result is a consequence of Example 10.5 of [5] and Section 4 in Chapter VIII of [3].

Proposition 2.2. *The natural action of $\mathrm{SU}(n+1)$ on S^{2n+1} induces a holomorphic action of $\mathrm{PSU}(n+1)$ on $\mathbb{P}^n(\mathbb{C})$ that satisfies the following properties:*

- *$\lambda(A)[w] = [Aw]$ for every $w \in S^{2n+1}$ and $A \in \mathrm{SU}(n+1)$.*
- *The induced $\mathrm{PSU}(n+1)$-action on $\mathbb{P}^n(\mathbb{C})$ realizes the connected component $\mathrm{Iso}_0(\mathbb{P}^n(\mathbb{C}))$ of the group of isometries of $\mathbb{P}^n(\mathbb{C})$ for the Fubini-Study metric.*

Hence, we have an isomorphism of Lie groups $\mathrm{Iso}_0(\mathbb{P}^n(\mathbb{C})) \simeq \mathrm{PSU}(n+1)$.

3. Quantum line bundles and quantization on compact Kähler manifolds

Let M be a Kähler manifold with Kähler form ω. We will now recall the notion of a quantum line bundle over M which allows to consider the Berezin-Toeplitz quantization for suitable sections over M. We refer to [1] and [11] for further details.

Suppose that $L \to M$ is a holomorphic line bundle. For any such line bundle, we will denote with $\Gamma(U, L)$ the space of smooth sections of L over an open set U of M. A Hermitian metric h on L is a smooth choice of Hermitian inner products on the fibers of L. For such metric, the pair (L, h) is called a Hermitian line bundle.

A connection D on L is given by assignments

$$D : \Gamma(U, L) \to \Gamma^1(U, L),$$

where U is an open subset of M and $\Gamma^k(U, L)$ denotes the space of L-valued k-forms over U; also, D must be complex linear and satisfy the following property

$$D(f\zeta) = df \otimes \zeta + fD\zeta,$$

for every $f \in C^\infty(U)$ and $\zeta \in \Gamma(U, L)$. We can extend D as a derivation on L-valued forms so that it defines maps $D : \Gamma^k(U, L) \to \Gamma^{k+1}(U, L)$. Then, the curvature of D is defined as $D^2 : \Gamma(U, L) \to \Gamma^2(U, L)$. It is well known that D^2 is linear with respect to the multiplication by smooth functions, which implies that D^2 defines a $L \otimes L^*$-valued 2-form. The latter is called the curvature of D.

Also, the connection D is said to be compatible with h if the following conditions are satisfied:

- For every $\zeta_1, \zeta_2 \in \Gamma(U, L)$ and X a smooth vector field over U we have
$$X(h(\zeta_1, \zeta_2)) = h(D_X\zeta_1, \zeta_2) + h(\zeta_1, D_X\zeta_2).$$
- For every holomorphic section $\zeta \in \Gamma(U, L)$ and X a smooth vector field over U of type $(0, 1)$ we have
$$D_X\zeta = 0.$$

The following result establishes the existence and uniqueness of compatible connections. It also provides an expression for the curvature form. We refer to [1] for its proof.

Proposition 3.1. *For a Hermitian line bundle (L, h) over a Kähler manifold M, there exist a unique connection D that is compatible with h. The curvature can be considered as a complex-valued $(1, 1)$-form Θ. Moreover, if ζ is a local holomorphic section of L on the open set U, then on this open set the curvature is given by $\Theta = \overline{\partial}\partial \log(h(\zeta, \zeta))$.*

The connection from Proposition 3.1 is called the Hermitian connection of the Hermitian line bundle.

A Hermitian line bundle (L, h) over a Kähler manifold M is called a quantum line bundle if it satisfies the condition

$$\Theta = -i\omega,$$

where Θ is the curvature of L and ω is the Kähler form of M. By Proposition 3.1, this is equivalent to requiring

$$\omega = i\overline{\partial}\partial \log(h(\zeta, \zeta)),$$

for every open set $U \subset M$ and every local holomorphic section of L defined on U.

For any Hermitian line bundle (L, h) and $m \in \mathbb{Z}_+$, we denote $L^m = L \otimes \cdots \otimes L$ (m times), which is itself a Hermitian line bundle with the induced metric. We will denote the latter by $h^{(m)}$. Furthermore, it is easy to see that the Hermitian connection of L^m is the induced connection of D to the tensor product. We will denote such connection by $D^{(m)}$.

We recall that the volume form of the Kähler manifold (M, ω) is given by $\Omega = \omega^n/(2\pi)^n$, where $n = \dim_{\mathbb{C}}(M)$. Note that for a quantum line bundle, this volume can be considered as coming from the geometry of either M or L.

In the rest of this section we assume that M is compact, and so that Ω has finite total volume. On each space $\Gamma(M, L^m)$ we define the Hermitian inner product

$$\langle \zeta, \xi \rangle = \int_M h^{(m)}(\zeta, \xi)\Omega,$$

where $\zeta, \xi \in \Gamma(M, L^m)$. The L_2-completion of the latter Hermitian space is denoted by $L_2(M, L^m)$. Since M is compact, the space of global holomorphic sections of L^m, denoted by $\Gamma_{hol}(M, L^m)$, is finite dimensional and so closed in $L_2(M, L^m)$. In particular, we have an orthogonal projection

$$\Pi_m : L_2(M, L^m) \to \Gamma_{hol}(M, L^m),$$

for every positive integer m.

4. Quantization and Bergman spaces on $\mathbb{P}^n(\mathbb{C})$

In this section we recollect some known properties and facts of the projective space $\mathbb{P}^n(\mathbb{C})$ that provide its quantization and the Bergman spaces on it. For further details on the less elementary facts we refer to [1] and [11].

Recall that the tautological or universal line bundle of $\mathbb{P}^n(\mathbb{C})$ is given by

$$T = \{([w], z) \in \mathbb{P}^n(\mathbb{C}) \times \mathbb{C}^{n+1} : z \in \mathbb{C}w\},$$

and assigns to every point in $\mathbb{P}^n(\mathbb{C})$ the line in $\mathbb{C}^{n+1}$ that such point represents. It is well known that T is a holomorphic line bundle. Furthermore, T has a natural Hermitian metric h_0 inherited from the usual Hermitian inner product on $\mathbb{C}^{n+1}$. Let us denote by $H = T^*$ the dual line bundle with the corresponding induced metric h dual to the metric h_0 on T. The line bundle H is called the hyperplane line bundle. The following well known result provides a quantization for $\mathbb{P}^n(\mathbb{C})$ as described in Section 3.

Proposition 4.1. *The Hermitian line bundle (H, h) is a quantum line bundle over $\mathbb{P}^n(\mathbb{C})$.*

For every $m \in \mathbb{Z}_+$ and with respect to the coordinates given by φ_0, the weighted measure on $\mathbb{P}^n(\mathbb{C})$ with weight m is given by

$$d\nu_m(z) = \frac{(n+m)!}{m!} \frac{\Omega(z)}{(1 + |z_1|^2 + \cdots + |z_n|^2)^m},$$

which has the following explicit expression

$$d\nu_m(z) = \frac{(n+m)!}{\pi^n m!} \frac{dV(z)}{(1 + |z_1|^2 + \cdots + |z_n|^2)^{n+m+1}}$$

where, as before, $dV(z) = dx_1 \wedge dy_1 \wedge \cdots \wedge dx_n \wedge dy_n$ is the Lebesgue measure on $\mathbb{C}^n$. For the sake of simplicity, we will use the same symbol $d\nu_m$ to denote the weighted

measures for both $\mathbb{P}^n(\mathbb{C})$ and $\mathbb{C}^n$. Note that that $d\nu_m$ is a probability measure. Following the remarks of Section 3, the Hilbert space $L_2(\mathbb{P}^n(\mathbb{C}), H^m)$ denotes the L_2-completion of $\Gamma(\mathbb{P}^n(\mathbb{C}), H^m)$ with respect to the inner product defined using the measure $d\nu_m$.

The line bundle H can be trivialized over each subset $U_j \subset \mathbb{P}^n(\mathbb{C})$ (U_j as in Section 2) so that the corresponding set of transition functions for H^m are given as follows

$$\begin{aligned} g^m_{kj} : U_j \cap U_k &\to \mathbb{C}^* \\ [w] &\mapsto \frac{w_k^m}{w_j^m}. \end{aligned}$$

In particular, there is a trivialization of H^m over U_0. Then, every section ζ of H^m restricted to U_0 can be considered as a map $\zeta|_{U_0} : U_0 \to \mathbb{C}$. Since $\varphi_0^{-1} : \mathbb{C}^n \to U_0$ defines a biholomorphism, the composition $\widehat{\zeta} = \zeta|_{U_0} \circ \varphi_0^{-1}$ maps $\mathbb{C}^n \to \mathbb{C}$. Then, we have the following well known result.

Proposition 4.2. *The map given by*

$$\begin{aligned} \Phi_0 : L_2(\mathbb{P}^n(\mathbb{C}), H^m) &\to L_2(\mathbb{C}^n, \nu_m) \\ \zeta &\mapsto \widehat{\zeta}, \end{aligned}$$

is an isometry of Hilbert spaces.

The weighted Bergman space on $\mathbb{P}^n(\mathbb{C})$ with weight $m \in \mathbb{Z}_+$ is defined by

$$\begin{aligned} \mathcal{A}^2_m(\mathbb{P}^n(\mathbb{C})) &= \{\zeta \in L_2(\mathbb{P}^n(\mathbb{C}), H^m) : \zeta \text{ is holomorphic}\} \\ &= \Gamma_{hol}(\mathbb{P}^n(\mathbb{C}), H^m). \end{aligned}$$

As remarked before, since $\mathbb{P}^n(\mathbb{C})$ is compact, the space $\mathcal{A}^2_m(\mathbb{P}^n(\mathbb{C}))$ is finite dimensional. Furthermore, the next well known result provides an explicit description of these Bergman spaces.

Proposition 4.3. *For every $m \in \mathbb{Z}_+$, the Bergman space $\mathcal{A}^2_m(\mathbb{P}^n(\mathbb{C}))$ satisfies the following properties.*

1. *With respect to the homogeneous coordinates of $\mathbb{P}^n(\mathbb{C})$, the Bergman space $\mathcal{A}^2_m(\mathbb{P}^n(\mathbb{C}))$ can be identified with the space $P^{(m)}(\mathbb{C}^{n+1})$ of homogeneous polynomials of degree m over $\mathbb{C}^{n+1}$.*
2. *For Φ_0 the isometry from Proposition 4.2, we have*

$$\Phi_0(\mathcal{A}^2_m(\mathbb{P}^n(\mathbb{C}))) = P_m(\mathbb{C}^n),$$

where $P_m(\mathbb{C}^n)$ denotes the space of polynomials on $\mathbb{C}^n$ of degree $\leq m$.

Propositions 4.2 and 4.3 allow us to reduce our computations on the spaces $L_2(\mathbb{P}^n(\mathbb{C}), H^m)$ and $\mathcal{A}^2_m(\mathbb{P}^n(\mathbb{C}))$ to the spaces $L_2(\mathbb{C}^n, \nu_m)$ and $P_m(\mathbb{C}^n)$, respectively. In what follows and when needed, we will use such reductions by applying the corresponding identifications without further mention.

The Bergman space $\mathcal{A}^2_m(\mathbb{P}^n(\mathbb{C}))$ identified with $P_m(\mathbb{C}^n)$ has the natural monomial basis which we will use for our computations. Such monomials are denoted by $z^p = z_1^{p_1} \ldots z_n^{p_n}$ where $p = (p_1, \ldots, p_n)$ and $|p| = p_1 + \cdots + p_n \leq m$. We denote the

enumerating set by $J_n(m) = \{p \in \mathbb{Z}_+^n : |p| \leq m\}$. A direct computation shows that

$$\begin{aligned}\langle z^p, z^p\rangle_m &= \frac{(n+m)!}{\pi^n m!}\int_{\mathbb{C}^n} \frac{z^p\overline{z}^p dV(z)}{(1+|z_1|^2+\cdots+|z_n|^2)^{n+m+1}} \\ &= \frac{(n+m)!}{m!}\int_{\mathbb{R}_+^n} \frac{t_1^{p_1}\cdots t_n^{p_n} dt_1\cdots dr_n}{(1+t_1+\cdots+t_n)^{n+m+1}} \\ &= \frac{p!(m-|p|)!}{m!}\end{aligned}$$

for every $p, q \in J_n(M)$, where $p! = p_1!\cdots p_n!$ and $p \in J_n(m)$. Also, it is easy to check that $\langle z^p, z^q\rangle_m = 0$ for all $p, q \in J_n(m)$ such that $p \neq q$. In particular, the set

$$\left\{\left(\frac{m!}{p!(m-|p|)!}\right)^{\frac{1}{2}} z^p : p \in J_n(m)\right\} \tag{4.1}$$

is an orthonormal basis of $\mathcal{A}_m^2(\mathbb{P}^n(\mathbb{C}))$.

The following result provides the classical description of the weighted Bergman projections of $L_2(\mathbb{P}^n(\mathbb{C}), H^m)$ onto $\mathcal{A}_m^2(\mathbb{P}^n(\mathbb{C}))$. In particular, these projections are precisely the orthogonal maps Π_m from Section 3 for $\mathbb{P}^n(\mathbb{C})$.

Proposition 4.4. *Let $B_m : L_2(\mathbb{P}^n(\mathbb{C}), H^m) \to L_2(\mathbb{P}^n(\mathbb{C}), H^m)$ be the operator given by the expression*

$$B_m(\psi)(z) = \frac{(n+m)!}{\pi^n m!}\int_{\mathbb{C}^n} \frac{\psi(w)K(z,w)dV(w)}{(1+|w_1|^2+\cdots+|w_n|^2)^{n+m+1}},$$

where $K(z,w) = (1+z_1\overline{w}_1\cdots+z_n\overline{w}_n)^m$. Then, B_m satisfies the following properties

1. *If $\psi \in L_2(\mathbb{P}^n(\mathbb{C}), H^m)$, then $B_m(\psi) \in \mathcal{A}_m^2(\mathbb{P}^n(\mathbb{C}))$.*
2. *$B_m(\psi) = \psi$ for every $\psi \in \mathcal{A}_m^2(\mathbb{P}^n(\mathbb{C}))$.*

In particular, B_m is the orthogonal projection $L_2(\mathbb{P}^n(\mathbb{C}), H^m) \to \mathcal{A}_m^2(\mathbb{P}^n(\mathbb{C}))$. Also, $K(z,w)$ is the Bergman kernel for $L_2(\mathbb{P}^n(\mathbb{C}), H^m)$.

5. Toeplitz operators with separately radial symbols

We introduce a decomposition for the projective space $\mathbb{P}^n(\mathbb{C})$ which is similar in spirit to the quasi-elliptic decomposition used for the n-dimensional unit ball in [8].

Consider the polar coordinates $z_j = t_j r_j$ where $t_j \in \mathbb{T}$ and $r_j \in \mathbb{R}_+$, for every $j = 1, \dots, n$. This yields, for all points $z \in \mathbb{C}^n$, an identification

$$z = (z_1, \dots, z_n) = (t_1 r_1, \dots, t_n r_n) = (t, r),$$

where $t = (t_1, \dots, t_n) \in \mathbb{T}^n$, $r = (r_1, \dots, r_n) \in \mathbb{R}_+^n$. In particular, we have $\mathbb{C}^n = \mathbb{T}^n \times \mathbb{R}_+^n$, with the corresponding volume form

$$dV(z) = \prod_{j=1}^n \frac{dt_j}{it_j}\prod_{j=1}^n r_j dr_j.$$

Hence, for the measure ν_m on $\mathbb{C}^n$ introduced in Section 4, we obtain the decomposition

$$L_2(\mathbb{C}^n, \nu_m) = L_2(\mathbb{T}^n) \otimes L_2(\mathbb{R}_+^n, \mu_m),$$

where

$$L_2(\mathbb{T}^n) = \bigotimes_{j=1}^n L_2\left(\mathbb{T}, \frac{dt_j}{2\pi i t_j}\right)$$

and the measure $d\mu_m$ of $L_2(\mathbb{R}^n_+, \mu_m)$ is given by

$$d\mu_m = \frac{(n+m)!}{m!}(1 + r_1^2 + \cdots + r_n^2)^{-n-m-1}\prod_{j=1}^n r_j dr_j.$$

We note that the Bergman space $\mathcal{A}^2_m(\mathbb{P}^n(\mathbb{C}))$ is given, in the local coordinates from φ_0, as the (closed) subspace of $L_2(\mathbb{C}^n, \nu_m)$ which consists of all functions satisfying the equations

$$\frac{\partial \varphi}{\partial \overline{z}_j} = \frac{1}{2}\left(\frac{\partial}{\partial x_j} + i\frac{\partial}{\partial y_j}\right)\varphi = 0, \quad j = 1, \dots, n,$$

or, in polar coordinates,

$$\frac{\partial \varphi}{\partial \overline{z}_j} = \frac{t_j}{2}\left(\frac{\partial}{\partial r_j} - \frac{t_j}{r_j}\frac{\partial}{\partial t_j}\right)\varphi = 0, \quad j = 1, \dots, n.$$

Now consider the discrete Fourier transform $\mathfrak{F} : L_2(\mathbb{T}) \to l_2 = l_2(\mathbb{Z})$ defined by

$$\mathfrak{F} : f \mapsto \left\{c_j = \int_{S^1} f(f)t^{-j}\frac{dt}{2\pi i t}\right\}_{j\in\mathbb{Z}},$$

so that, in particular, the operator $\mathfrak{F}$ is unitary with inverse given by

$$\mathfrak{F}^{-1} = \mathfrak{F}^* : \{c_j\}_{j\in\mathbb{Z}} \mapsto f = \sum_{j\in\mathbb{Z}} c_j t^j$$

Now, let us consider the operator $u : l_2 \otimes L_2((0,1), rdr) \to l_2 \otimes L_2((0,1), rdr)$ given by the composition

$$u = (\mathfrak{F} \otimes I)\frac{t}{2}\left(\frac{\partial}{\partial r} - \frac{t}{r}\frac{\partial}{\partial t}\right)(\mathfrak{F}^{-1} \otimes I).$$

Then, it is easy to check (see Subsection 4.1 of [12]) that u acts by

$$\{c_j(r)\}_{j\in\mathbb{Z}} \mapsto \left\{\frac{1}{2}\left(\frac{\partial}{\partial r} - \frac{j}{r}\right)c_j(r)\right\}_{j\in\mathbb{Z}}$$

Introduce the unitary operator

$$U = \mathfrak{F}_{(n)} \otimes I : L_2(\mathbb{T}^n) \otimes L_2(\mathbb{R}^n_+, \nu_m) \to l_2(\mathbb{Z}^n) \otimes L_2(\mathbb{R}^n_+, \nu_m)$$

where $\mathfrak{F}_{(n)} = \mathfrak{F} \otimes \cdots \otimes \mathfrak{F}$ (n times). Then, the image $\mathcal{A}^2_m = U(\mathcal{A}^2_m(\mathbb{P}^n(\mathbb{C})))$ under U of the Bergman space is the closed subspace of $l_2(\mathbb{Z}^n) \otimes L_2(\mathbb{R}^n_+, \nu_m)$ which consist of all sequences $\{c_p(r)\}_{p\in\mathbb{Z}^n}$, $r = (r_1, \dots, r_n) \in \mathbb{R}^n_+$, satisfying the equations

$$\frac{1}{2}\left(\frac{\partial}{\partial r_j} - \frac{p_j}{r_j}\right)c_p(r) = 0,$$

for all $p_j \in \mathbb{Z}$ and $j = 1, \dots, n$. The general solution of this system of equations has the form

$$c_p(r) = \alpha_p^m c_p r^p,$$

for all $p \in \mathbb{Z}^n$, where $c_p \in \mathbb{C}$, $r^p = r_1^{p_1} \cdots r_n^{p_n}$ and $\alpha_p = \alpha_{(|p_1|,\dots,|p_n|)}$ is given by

$$\begin{aligned}
\alpha_p^m &= \left(\frac{(n+m)!}{m!}\int_{\mathbb{R}^n_+} \frac{t_1^{p_1}\cdots t_n^{p_n} t_1 dt_1 \cdots t_n d_n}{(1 + t_1 + \cdots + t_n)^{n+m+1}}\right)^{-\frac{1}{2}} \\
&= \left(\frac{m!}{p!(m - |p|)!}\right)^{\frac{1}{2}}
\end{aligned}$$

We have that every function $c_p(r) = \alpha_p^m c_p r^p$ has to be in $L_2(\mathbb{R}_+^n, \nu_m)$, and this integrability condition implies $c_p = 0$ for every $p \in \mathbb{Z}^n \setminus J_n(m)$. Hence, $\mathcal{A}_m^2 \subset l_2(\mathbb{Z}^n) \otimes L_2(\mathbb{R}_+^n, \nu_m)$ coincides with the space of all sequences that satisfy

$$c_p(r) = \begin{cases} \left(\frac{m!}{p!(m-|p|)!}\right)^{\frac{1}{2}} c_p r^p & p \in J_n(m) \\ 0 & p \in \mathbb{Z}^n \setminus J_n(m) \end{cases}, \tag{5.1}$$

and for all such sequences we have

$$\|\{c_p(r)\}_{p\in J_n}\|_{l_2(\mathbb{Z}^n)\otimes L_2(\mathbb{R}_+^n,\nu_m)} = \|\{c_p\}_{p\in J_n}\|_{l_2(\mathbb{Z}^n)}.$$

In particular, we have a natural isometric identification $\mathcal{A}_m^2 = l_2(J_n(m))$. These constructions allow us to consider the isometric embedding

$$R_0 : l_2(J_n(m)) \to l_2(\mathbb{Z}^n) \otimes L_2(\mathbb{R}_+^n, \nu_m)$$

defined by

$$R_0 : \{c_p\}_{p\in J_n(m)} \mapsto c_p(r) = \begin{cases} \left(\frac{m!}{p!(m-|p|)!}\right)^{\frac{1}{2}} c_p r^p & p \in J_n(m) \\ 0 & p \in \mathbb{Z}^n / J_n(m) \end{cases}.$$

For which the adjoint operator $R_0^* : l_2(\mathbb{Z}^n) \otimes L_2(\mathbb{R}_+^n, \nu_m) \to l_2(J_n(m))$ is given by

$$R_0^* : \{f_p(r)\}_{p\in\mathbb{Z}^n} \mapsto \left\{ \left(\frac{m!}{p!(m-|p|)!}\right)^{\frac{1}{2}} \int_{\mathbb{R}_+^n} r^p f_p(r) d\nu_m(r) \right\}_{p\in J_n(m)}.$$

It is easily seen that

$$\begin{aligned} R_0^* R_0 = I \quad &: \quad l_2(J_n(m)) \to l_2(J_n(m)) \\ R_0 R_0^* = P_1 \quad &: \quad l_2(\mathbb{Z}^n) \otimes L_2(\mathbb{R}_+^n, \nu_m) \to l_2(J_n(m)) \end{aligned}$$

where P_1 is the ortogonal projection of $l_2(\mathbb{Z}^n) \otimes L_2(\mathbb{R}_+^n, \nu_m)$ onto $l_2(J_n(M)) = \mathcal{A}_m^2$. Summarizing the above we have the next result.

Theorem 5.1. *The operator $R = R_0^* U$ maps $L_2(\mathbb{P}^n(\mathbb{C}), \nu_m)$ onto $\mathcal{A}_m^2 = l_2(J_n(m))$, and its restriction*

$$R|_{\mathcal{A}_m^2(\mathbb{P}^n(\mathbb{C}))} : \mathcal{A}_m^2(\mathbb{P}^n(\mathbb{C})) \to l_2(J_n(m))$$

is an isometric isomorphism. The adjoint operator

$$R^* = U^* R_0 : l_2(J_n(m)) \to \mathcal{A}_m^2(\mathbb{P}^n(\mathbb{C})) \subset L_2(\mathbb{P}^n(\mathbb{C}), \nu_m)$$

is an isometric isomorphism of $l_2(J_n(m)) = \mathcal{A}_m^2$ onto the subspace $\mathcal{A}_m^2(\mathbb{P}^n(\mathbb{C}))$. Furthermore

$$\begin{gathered} RR^* = I : l_2(J_n(m)) \to l_2(J_n(m)) \\ R^* R = B_m : L_2(\mathbb{P}^n(\mathbb{C}), \nu_m) \to \mathcal{A}_m^2(\mathbb{P}^n(\mathbb{C})) \end{gathered}$$

where B_m is the Bergman projection of $L_2(\mathbb{P}^n(\mathbb{C}), \nu_m)$ onto $\mathcal{A}_m^2(\mathbb{P}^n(\mathbb{C}))$

We note that an explicit calculation yields

$$\begin{aligned} R^* = U^*R_0 : \{c_p\}_{p\in J_n(m)} \mapsto U^* &\left(\left\{ \left(\frac{m!}{p!(m-|p|)!} \right)^{\frac{1}{2}} c_p r^p \right\}_{p\in J_n(m)} \right) \\ &= \sum_{p\in J_n(m)} \left(\frac{m!}{p!(m-|p|)!} \right)^{\frac{1}{2}} c_p (rt)^p \\ &= \sum_{p\in J_n(m)} \left(\frac{m!}{p!(m-|p|)!} \right)^{\frac{1}{2}} c_p z^p, \end{aligned}$$

which implies the following result.

Corollary 5.2. *With the notation of Theorem 5.1, the isometric isomorphism $R^* : l_2(J_n(m)) \to \mathcal{A}^2_m(\mathbb{P}^n(\mathbb{C})) \subset L_2(\mathbb{P}^n(\mathbb{C}), \nu_m)$ is given by*

$$R^* : \{c_p\}_{p\in J_n(m)} \mapsto \sum_{p\in J_n(m)} \left(\frac{m!}{p!(m-|p|)!} \right)^{\frac{1}{2}} c_p z^p. \tag{5.2}$$

A similar direct computation yields the following result.

Corollary 5.3. *With the notation of Theorem 5.1, the isometric isomorphism $R|_{\mathcal{A}^2_m(\mathbb{P}^n(\mathbb{C}))} : \mathcal{A}^2_m(\mathbb{P}^n(\mathbb{C})) \to l_2(J_n(m))$ is given by*

$$R : \psi \mapsto \left\{ \left(\frac{m!}{p!(m-|p|)!} \right)^{\frac{1}{2}} \int_{\mathbb{C}^n} \psi(z)\overline{z}^p d\nu_m(z) \right\}_{p\in J_n(m)}. \tag{5.3}$$

We now introduce a special family of symbols on $\mathbb{C}^n$.

Definition 5.4. We will call a function $a(z)$, $z \in \mathbb{C}^n$ separately radial if $a(z) = a(r)$, i.e. a depends only on the radial components of $z = (t, r)$.

The separately radial symbols give rise to a C^*-algebra of Toeplitz operators that can be turned simultaneously into multiplication operators.

Theorem 5.5. *Let a be a bounded measure separately radial function. Then the Toeplitz operator T_a acting on $\mathcal{A}^2_m(\mathbb{P}^n(\mathbb{C}))$ is unitary equivalent to the multiplication operator $\gamma_{a,m}I = RT_aR^*$ acting on $l_2(J_n(m))$, where R and R^* are given by* (5.3) *and* (5.2) *respectively. The sequence $\gamma_{a,m} = \{\gamma_{a,m}(p)\}_{p\in J_n(m)}$ is explicitely given by*

$$\gamma_{a,m}(p) = \frac{2^n m!}{p!(m-|p|)!} \int_{\mathbb{R}^n_+} \frac{a(r_1, \ldots, r_n) r_1^{2p_1+1} \cdots r_n^{2p_n+1} dr_1 \cdots dr_n}{(1 + r_1^2 + \cdots + r_n^2)^{n+m+1}}. \tag{5.4}$$

Proof. Using the previous results, the operator T_a is unitary equivalent to the operator

$$\begin{aligned} RT_aR^* &= RB_m a B_m R^* = R(R^*R)a(R^*R)R^* \\ &= (RR^*)RaR^*(RR^*) = RaR^* \\ &= R_0^* U a U^* R_0 \\ &= R_0^*(\mathfrak{F}_{(n)} \otimes I) a (\mathfrak{F}^{-1}_{(n)} \otimes I) R_0 \\ &= R_0^* a R_0. \end{aligned}$$

And the latter operator is computed as follows

$$
\begin{aligned}
& R_0^* a(r) R_0 \{c_p\}_{p\in J_n(m)} \\
&= R_0^* \left\{ \left(\frac{m!}{p!(m-|p|)!} \right)^{\frac{1}{2}} a(r) c_p r^p \right\}_{p\in J_n(m)} \\
&= \left\{ \left(\frac{m!}{p!(m-|p|)!} \right)^{\frac{1}{2}} \int_{\mathbb{R}_+^n} \frac{a(r) c_p r^{2p} r dr}{(1+r_1^2+\cdots+r_n^2)^{n+m+1}} \right\}_{p\in J_n(m)} \\
&= \{\gamma_{a,m}(p)\cdot c_p\}_{p\in J_n(m)}.
\end{aligned}
$$

□

Hence we can diagonalize the corresponding Toeplitz operators.

Corollary 5.6. *The Toeplitz operator T_a with bounded measurable separately radial symbol $a(r)$ is diagonal with respect to the orthonormal base given by* (4.1)*. More precisely, we have*

$$T_a\left(\left(\frac{m!}{p!(m-|p|)!} \right)^{\frac{1}{2}} z^p \right) = \gamma_{a,m}(p) \left(\frac{m!}{p!(m-|p|)!} \right)^{\frac{1}{2}} z^p$$

for all $p\in J_n(m)$.

6. Abelian groups and Lagrangian frames on $\mathbb{P}^n(\mathbb{C})$

In this section, we will discuss the geometric features of separately radial symbols. Such description will be provided in the context of both actions of Lie groups and foliated spaces.

Let us denote with $\widehat{\mathbb{A}}(n)$ the subgroup of diagonal matrices in the Lie group $\mathrm{SU}(n+1)$, which is obviously a connected Abelian Lie subgroup of dimension n. Correspondingly, we denote $\mathbb{A}(n) = \lambda(\widehat{\mathbb{A}}(n))$, which is a connected Abelian Lie subgroup of $\mathrm{PSU}(n+1)$ of dimension n. Recall from Section 2 that λ denotes the natural projection map $\mathrm{SU}(n+1) \to \mathrm{PSU}(n+1)$. The group $\mathbb{A}(n)$ describes our separately radial symbols by use of the next result, whose proof follows readly from the definitions involved.

Lemma 6.1. *Let $\varphi_0 : U_0 \to \mathbb{C}^n$ be the coordinate chart of $\mathbb{P}^n(\mathbb{C})$ defined in Section 2. Then, U_0 is $\mathbb{A}(n)$-invariant and induces on $\mathbb{C}^n$ the $\mathbb{T}^n$-action given by*

$$
\begin{aligned}
\mathbb{T}^n \times \mathbb{C}^n &\to \mathbb{C}^n \\
(t,z) &\mapsto tz = (t_1 z_1, \dots, t_n z_n).
\end{aligned}
$$

More precisely, if we let $\rho : \mathbb{T}^n \to \widehat{\mathbb{A}}(n)$ be the isomorphism defined by

$$\rho(t) = \begin{pmatrix} t_1 & 0 & \dots & 0 & 0 \\ 0 & t_2 & \dots & 0 & 0 \\ \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & 0 & \dots & t_n & 0 \\ 0 & 0 & \dots & 0 & \overline{t}_1 \dots \overline{t}_n \end{pmatrix}$$

then, we have

$$t\varphi_0([w]) = \varphi_0([\rho(t)w]),$$

for every $t \in \mathbb{T}^n$ and $[w] \in U_0$. In particular, a symbol $a : \mathbb{C}^n \to \mathbb{C}$ is separately radial if and only if $a \circ \varphi_0$ is $\mathbb{A}(n)$-invariant as a function on $U_0 \subset \mathbb{P}^n(\mathbb{C})$.

Note that a change of coordinates by a biholomorphism of $\mathbb{P}^n(\mathbb{C})$ can be used to obtain from our separately radial symbols an equivalent family of C^*-algebras of Toeplitz operators. Correspondingly, conjugation by any such biholomorphism provides an Abelian group equivalent to $\mathbb{A}(n)$. In our search for commutative C^*-algebras of Toeplitz operators this suggests to study the relation of $\mathbb{A}(n)$ with other possible Abelian groups, particularly those obtained by conjugations. The following result provides a complete solution to such problem within the group $\mathrm{PSU}(n+1)$ that defines the connected component $\mathrm{Iso}_0(\mathbb{P}^n(\mathbb{C}))$ of the group of isometries of $\mathbb{P}^n(\mathbb{C})$.

Theorem 6.2. *The Lie subgroup $\mathbb{A}(n)$ of $\mathrm{PSU}(n+1)$ satisfies the following properties.*

1. *$\mathbb{A}(n)$ is isomorphic to $\mathbb{T}^n$.*
2. *$\mathbb{A}(n)$ is a maximal Abelian subgroup (MASG) of $\mathrm{PSU}(n+1)$; i.e. if H is a connected Abelian proper subgroup of $\mathrm{PSU}(n+1)$ that contains $\mathbb{A}(n)$, then $H = \mathbb{A}(n)$.*
3. *If H is a connected Abelian Lie subgroup of $\mathrm{PSU}(n+1)$, then there exist $g \in \mathrm{PSU}(n+1)$ such that $gHg^{-1} \subset \mathbb{A}(n)$.*

Proof. First note that an isomorphism between $\mathbb{A}(n)$ and $\mathbb{T}^n$ was already provided in Lemma 6.1.

The rest of the statement follows from the fact that $\mathrm{PSU}(n+1)$ is a compact connected group of matrices. We explain how to prove the remaining claims from the results found in [4].

By the well known correspondence between Lie subalgebras and Lie subgroups, we conclude that the MASG's of $\mathrm{PSU}(n+1)$ are precisely those whose Lie algebra are maximal Abelian subalgebras of the Lie algebra of $\mathrm{PSU}(n+1)$. Hence, it follows from Proposition 4.30 of [4] that the MASG are precisely the maximal tori in $\mathrm{PSU}(n+1)$. We recall that a torus is a group isomorphic to a product of circles, and that the maximality here refers to the order by inclusion. Hence, Example 1 in page 252 of [4] implies that $\mathbb{A}(n)$ is a MASG of $\mathrm{PSU}(n+1)$, which yields (2).

Finally, the last assertion of the statement follows from Corollary 4.35 of [4] and our remark above on the fact that the MASG's in $\mathrm{PSU}(n+1)$ are precisely the maximal tori. $\square$

As a consequence, we obtain the following uniqueness result for the isometry group $\mathrm{Iso}_0(\mathbb{P}^n(\mathbb{C}))$. This follows from Theorem 6.2 and Proposition 2.2.

Theorem 6.3. *A subgroup H of $\mathrm{Iso}_0(\mathbb{P}^n(\mathbb{C}))$ is a MASG if and only if there exist $\varphi \in \mathrm{Iso}_0(\mathbb{P}^n(\mathbb{C}))$ such that $\varphi H \varphi^{-1} = \mathbb{A}(n)$.*

To continue the description of our separately radial symbols on $\mathbb{P}^n(\mathbb{C})$ we will use the notion of a foliation and its geometric properties.

We recall the definition of a foliation as found in [9]. On a smooth manifold M a codimension q foliated chart is a pair (φ, U) given by an open subset U of M and a smooth submersion $\varphi : U \to V$, where V is an open subset of $\mathbb{R}^q$. For a foliated chart (φ, U) the connected components of the fibers of φ are called the plaques of the foliated chart. Two codimension q foliated charts (φ_1, U_1) and (φ_2, U_2) are

called compatible if there exists a diffeomorphism $\psi_{12} : \varphi_1(U_1 \cap U_2) \to \varphi_2(U_1 \cap U_2)$ such that the following diagram commutes

$$\begin{tikzcd}[column sep=small] & U_1 \cap U_2 \arrow[dl, "\varphi_1"'] \arrow[dr, "\varphi_2"] & \\ \varphi_1(U_1 \cap U_2) \arrow[rr, "\psi_{12}"] & & \varphi_2(U_1 \cap U_2). \end{tikzcd} \tag{6.1}$$

A foliated atlas on a manifold M is a collection $\{(\varphi_\alpha, U_\alpha)\}_\alpha$ of foliated charts that are mutually compatible and such that $M = \bigcup_\alpha U_\alpha$.

The compatibility of two foliated charts (φ_1, U_1) and (φ_2, U_2) is defined so that it ensures that, when restricted to $U_1 \cap U_2$, both submersions φ_1 and φ_2 have the same plaques. This implies that the following is an equivalence relation in M.

$$x \sim y \quad \Longleftrightarrow \quad \begin{array}{l} \text{there is a sequence of plaques } (P_k)_{k=0}^l \text{ for foliated charts} \\ (\varphi_k, U_k)_{k=0}^l, \text{respectively, of the foliated atlas, such that} \\ x \in P_0,\ y \in P_l, \text{ and } P_{k-1} \cap P_k \neq \phi \text{ for every } k = 1, \dots, l \end{array}$$

The equivalence classes are submanifolds of M of dimension $\dim(M) - q$.

Definition 6.4. A foliation $\mathfrak{F}$ on a manifold M is a partition of M that is given by the family of equivalence classes of the relation of a foliated atlas. The classes are called the leaves of the foliation.

If $\mathfrak{F}$ is a foliation, we denote with $T\mathfrak{F}$ the space of tangent vectors to the leaves of $\mathfrak{F}$. The space $T\mathfrak{F}$ is called the tangent bundle of the foliation. We note that $T\mathfrak{F}$ is a vector subbundle of the tangent bundle to the ambient manifold. If E is a vector subbundle of the tangent bundle of the ambient manifold, then we will say that E is integrable if it is the tangent bundle of a foliation. We observe that not every vector bundle is integrable. Also, a vector bundle E is integrable if and only if through every point in the ambient manifold there is a submanifold N such that $TN_p = E_p$ for every $p \in N$.

For the geometric description of our separately radial symbols we need to consider Lagrangian foliations, i.e. those for which the leaves are Lagrangian submanifolds. We recall that a submanifold N of a symplectic manifold M is called Lagrangian if the tangent space T_pN is a Lagrangian subspace of T_pM for every $p \in N$.

We will also consider further properties for submanifolds which we collect in the next definition. We refer to [5] for more details.

Definition 6.5. Let M be a Riemannian manifold with Levi-Civita connection ∇, and N a submanifold of M.

(1) N is called complete if every geodesic in N can be defined for every value of $\mathbb{R}$ so that it still lies in N.
(2) N is called flat if it has vanishing sectional curvature for the metric inherited from M.
(3) N is called parallel if $\nabla \Pi = 0$, where Π is the second fundamental form of N with respect to M.
(4) N is called totally geodesic if every geodesic in N is a geodesic in M as well.

Given these definitions, the following provides the main object that will be used to geometrically describe the separately radial symbols on $\mathbb{P}^n(\mathbb{C})$.

Definition 6.6. A Lagrangian frame on an open connected subset D of $\mathbb{P}^n(\mathbb{C})$ is a pair of foliations $(\mathfrak{F}_1, \mathfrak{F}_2)$ that satisfy the following properties:

(1) The leaves of $\mathfrak{F}_1$ are complete flat parallel Lagrangian submanifolds.
(2) The leaves of $\mathfrak{F}_2$ are totally geodesic Lagrangian submanifolds.
(3) At their intersection, every leaf of $\mathfrak{F}_1$ is perpendicular to every leaf of $\mathfrak{F}_2$.

It has been shown that Lagrangian frames on the n-dimensional complex unit ball provide the natural geometric setup to study symbols generating commutative C^*-algebras of Toeplitz operators (see [8] and [9]). In our case of the separately radial symbols for $\mathbb{P}^n(\mathbb{C})$ we will show that a similar phenomenon takes place. We show that the geometry of the level sets of separately radial symbols yield a Lagrangian frame.

First, we present the full description of the complete flat parallel submanifolds of $\mathbb{P}^n(\mathbb{C})$ as found in [6].

Let us denote with $S^1(r)$ the circle in $\mathbb{C}$ with radius $r > 0$ and centered at the origin. For every $r \in S^n \cap \mathbb{R}_+^{n+1}$, the torus

$$\widehat{M}(r) = S^1(r_1) \times \cdots \times S^1(r_{n+1}) \subset \mathbb{C}^{n+1}$$

is clearly contained in S^{2n+1}. As above, we let $\pi : S^{2n+1} \to \mathbb{P}^n(\mathbb{C})$ denote the Hopf fibration of $\mathbb{P}^n(\mathbb{C})$. We will also denote $M(r) = \pi(\widehat{M}(r))$ for every $r \in S^n \cap \mathbb{R}_+^{n+1}$. The following result is a consequence of Theorems 2.1 and 3.1 from [6].

Theorem 6.7. *For every $r \in S^n \cap \mathbb{R}_+^{n+1}$, the set $M(r)$ is a connected complete flat parallel Lagrangian submanifold of $\mathbb{P}^n(\mathbb{C})$. Furthermore, if M is any connected complete flat parallel Lagrangian submanifold of $\mathbb{P}^n(\mathbb{C})$, then there exists $\varphi \in \mathrm{Iso}_0(\mathbb{P}^n(\mathbb{C}))$ such that $\varphi(M) = M(r)$ for some $r \in S^n \cap \mathbb{R}_+^{n+1}$.*

Let us denote

$$\mathbb{P}^n(\mathbb{C})_0 = \{[z_0, \dots, z_n] \in \mathbb{P}^n(\mathbb{C}) : z_j \neq 0 \text{ for every } j = 0, \dots, n\}.$$

Clearly, $\mathbb{P}^n(\mathbb{C})_0$ is a connected open subset of $\mathbb{P}^n(\mathbb{C})$ which is conull and dense as well. Also, the action of $\mathbb{A}(n)$ clearly preserves $\mathbb{P}^n(\mathbb{C})_0$ and restricted to this set it is free. In other words, if for some $p \in \mathbb{P}^n(\mathbb{C})_0$ and $g \in \mathbb{A}(n)$ we have $gp = p$, then $g = e$ (the identity element).

The following result provides a characterization of the flat parallel Lagrangian submanifolds in terms of the MASG $\mathbb{A}(n)$ of $\mathrm{Iso}_0(\mathbb{P}^n(\mathbb{C}))$.

Theorem 6.8. *If $p \in \mathbb{P}^n(\mathbb{C})_0$, then the orbit $\mathbb{A}(n)p$ is a connected complete flat parallel Lagrangian submanifold of $\mathbb{P}^n(\mathbb{C})$. Conversely, for every connected complete flat parallel Lagrangian submanifold M of $\mathbb{P}^n(\mathbb{C})$ there exists $p \in \mathbb{P}^n(\mathbb{C})_0$ such that $M = \mathbb{A}(n)p$.*

Proof. Let $p \in \mathbb{P}^n(\mathbb{C})_0$ be given and let $r_j = |z_j|$, for $j = 0, \dots, n$, where $p = [z_0, \dots, z_n]$. Without loss of generality, we can assume $z \in S^{2n+1}$ so that in particular $r = (r_0, \dots, r_n) \in S^n \cap \mathbb{R}_+^{n+1}$.

By the definition of $\mathbb{A}(n)$ and the $\mathrm{PSU}(n+1)$-action on $\mathbb{P}^n(\mathbb{C})$, it follows that

$$\mathbb{A}(n)p = \{[t_0 r_0, \dots, t_n r_n] : t_j \in S^1, j = 0, \dots, n\}.$$

Hence, we have $\mathbb{A}(n)p = M(r)$ and so Theorem 6.7 implies that $\mathbb{A}(n)p$ is a connected complete flat parallel Lagrangian submanifold of $\mathbb{P}^n(\mathbb{C})$.

Let us now assume that M is a connected complete flat parallel Lagrangian submanifold of $\mathbb{P}^n(\mathbb{C})$. By Theorem 6.7 there exist $\varphi \in \mathrm{Iso}_0(\mathbb{P}^n(\mathbb{C}))$ and $r \in$

$S^n \cap \mathbb{R}_+^{n+1}$ such that $\varphi(M) = M(r)$. But the above computation shows that $M(r) = \mathbb{A}(n)p$ for $p = [r_0, \dots, r_n] \in \mathbb{P}^n(\mathbb{C})$, thus completing the proof. □

Let us denote with $\mathcal{O}$ the set of $\mathbb{A}(n)$-orbits in $\mathbb{P}^n(\mathbb{C})_0$. The next result shows that $\mathcal{O}$ allows to obtain a Lagrangian frame of $\mathbb{P}^n(\mathbb{C})$ defined on $\mathbb{P}^n(\mathbb{C})_0$.

Theorem 6.9. *The collection $\mathcal{O}$ of $\mathbb{A}(n)$-orbits on $\mathbb{P}^n(\mathbb{C})_0$ satisfies the following properties:*

1. *$\mathcal{O}$ is a foliation whose leaves are complete flat parallel Lagrangian submanifolds of $\mathbb{P}^n(\mathbb{C})$.*
2. *The orthogonal complement $T\mathcal{O}^\perp$ of $T\mathcal{O}$ is integrable and the leaves of the associated foliation $\mathcal{O}^\perp$ are totally geodesic Lagrangian submanifolds of $\mathbb{P}^n(\mathbb{C})$.*

In particular, the pair $(\mathcal{O}, \mathcal{O}^\perp)$ defines a Lagrangian frame on the open subset $\mathbb{P}^n(\mathbb{C})_0$ of $\mathbb{P}^n(\mathbb{C})$.

Proof. We already noted that the $\mathbb{A}(n)$-action is free and preserves the Riemannian metric of $\mathbb{P}^n(\mathbb{C})$. Hence, $\mathcal{O}$ defines a foliation by Proposition 6.7 from [9]. Furthermore, by Theorem 6.8 the leaves of $\mathcal{O}$ are complete flat parallel Lagrangian submanifolds. This establishes (1).

Next we prove that $T\mathcal{O}^\perp$ is integrable. Since the leaves of $\mathcal{O}$ are Lagrangian, so are the fibres of the vector bundle $T\mathcal{O}^\perp$. In particular, we have $T\mathcal{O}^\perp = iT\mathcal{O}$. For a given $p \in \mathbb{P}^n(\mathbb{C})_0$ let us consider the space

$$N(p) = \{[r_0 z_0, \dots, r_n z_n] : r_j \in \mathbb{R}_+ \text{ for every } j = 0, \dots, n\}$$

where $p = [z_0, \dots, z_n]$. It is clear that $N(p)$ is a submanifold of $\mathbb{P}^n(\mathbb{C})$ and that $T_q N(p) = iT_q\mathcal{O}$ for every $q \in N$. This proves the integrability of $T\mathcal{O}^\perp$.

Finally, that the leaves of $\mathcal{O}^\perp$ are totally geodesic is a direct consequence of Proposition 6.9 from [9]. □

The following result shows that every Lagrangian frame is, up to an isometry, the Lagrangian frame $(\mathcal{O}, \mathcal{O}^\perp)$ defined above, and thus it is given by our separately radial symbols.

Theorem 6.10. *If $(\mathfrak{F}_1, \mathfrak{F}_2)$ is a Lagrangian frame defined in an open connected subset U of $\mathbb{P}^n(\mathbb{C})$, then there exist $\varphi \in \mathrm{Iso}_0(\mathbb{P}^n(\mathbb{C}))$ of $\mathbb{P}^n(\mathbb{C})$ such that:*

1. *$\varphi(U) \subset \mathbb{P}^n(\mathbb{C})_0$.*
2. *Every leaf of $\varphi(\mathfrak{F}_1)$ is a leaf of $\mathcal{O}$.*
3. *Every leaf of $\varphi(\mathfrak{F}_2)$ is an open subset of a leaf of $\mathcal{O}^\perp$.*

Proof. Let L be a leaf of $\mathfrak{F}_1$. Hence, L is a connected complete flat parallel Lagrangian submanifold of $\mathbb{P}^n(\mathbb{C})$. By Theorem 6.8, there exist $\varphi \in \mathrm{Iso}_0(\mathbb{P}^n(\mathbb{C}))$ and $p \in \mathbb{P}^n(\mathbb{C})_0$ such that $\varphi(L) = \mathbb{A}(n)p$. Let us consider the image under the exponential map exp of the normal bundle N to $\mathbb{A}(n)p$. Then, for every $q \in \mathbb{A}(n)p$ the set $\exp(N_q)$ is the largest totally geodesic submanifold of $\mathbb{P}^n(\mathbb{C})$ perpendicular to $\mathbb{A}(n)p$ at q. Since $(\mathfrak{F}_1, \mathfrak{F}_2)$ is a Lagrangian frame, it follows that for L' the leaf of $\mathfrak{F}_2$ through $\varphi^{-1}(q)$ we have

$$\varphi(L') \subset \exp(N_q).$$

Since $(\mathcal{O}, \mathcal{O}^\perp)$ is a Lagrangian frame as well, we conclude that $\exp(N_q)$ is a leaf of $\mathcal{O}^\perp$. This proves (3). But then (2) follows since the leaves of $\mathfrak{F}_1$ (resp. $\mathcal{O}$) are the integral submanifolds of the orthogonal complement of $T\mathfrak{F}_1$ (resp. $T\mathcal{O}$).

Finally (1) follows from the fact that U is the union of the leaves of $\mathfrak{F}_1$. □

We use the previous results to prove that every family of symbols associated to a Lagrangian frame is, up to a biholomorphism, a subset of our separately radial symbols.

Theorem 6.11. *For a subspace $\mathcal{A}$ of $C^\infty(\mathbb{P}^n(\mathbb{C}))$ the following conditions are equivalent.*

1. *There is a Lagrangian frame $(\mathfrak{F}_1, \mathfrak{F}_2)$ defined in a connected open conull subset U of $\mathbb{P}^n(\mathbb{C})$ such that if $a \in \mathcal{A}$, then every level set of a is saturated with respect to the foliation $\mathfrak{F}_1$, i.e., every such level set is a union of leaves of $\mathfrak{F}_1$.*
2. *There exist $\varphi \in \mathrm{Iso}_0(\mathbb{P}^n(\mathbb{C}))$ such that $\mathcal{A} \subset \varphi^*(\mathcal{A}_{\mathbb{A}(n)}) = \{a \circ \varphi : a \in \mathcal{A}_{\mathbb{A}(n)}\}$, where $\mathcal{A}_{\mathbb{A}(n)}$ is the subspace of $C^\infty(\mathbb{P}^n(\mathbb{C}))$ consisting of $\mathbb{A}(n)$-invariant functions.*

Proof. That (2) implies (1) is the content of Theorem 6.9 together with (the obvious) invariance of Lagrangian frames with respect to elements in $\mathrm{Iso}_0(\mathbb{P}^n(\mathbb{C}))$.

To prove that (1) implies (2) we use Theorem 6.10. From this result it follows that there exist $\varphi \in \mathrm{Iso}_0(\mathbb{P}^n(\mathbb{C}))$ such that the Lagrangian frame $(\mathfrak{F}_1, \mathfrak{F}_2)$ is mapped under φ^{-1} to a restriction of $(\mathcal{O}, \mathcal{O}^\perp)$. Hence, for every $a \in \mathcal{A}$, the level subsets of $a \circ \varphi$ are saturated with respect to the leaves of the foliation $\mathcal{O}$ on $\varphi^{-1}(U)$. This implies that, for every such a, the function $a \circ \varphi$ is $\mathbb{A}(n)$-invariant on $\varphi^{-1}(U)$. Hence, the result follows by the density of U that comes from the fact that it is conull. □

Corollary 6.12. *Given any Lagrangian frame $\mathfrak{F} = (\mathfrak{F}_1, \mathfrak{F}_2)$ on $\mathbb{P}^n(\mathbb{C})$, denote by $\mathcal{A}_{\mathfrak{F}}$ the set of all functions in $C^\infty(\mathbb{P}^n(\mathbb{C}))$ which are constant on the leaves of $\mathfrak{F}_1$. Then the C^*-algebra $\mathcal{T}_{\mathfrak{F}}$ generated by the Toeplitz operators with symbols in $\mathcal{A}_{\mathfrak{F}}$ is commutative on each weighted Bergman space $\mathcal{A}^2_m(\mathbb{P}^n(\mathbb{C}))$, $m \in \mathbb{Z}_+$. Furthermore, $\mathcal{T}_{\mathfrak{F}}$ is unitarily equivalent to $\mathcal{T}_{(\mathcal{O},\mathcal{O}^\perp)}$.*

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